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CRITICAL PATH ANALYSES VIA CHANCE CONSTRAINED AND STOCHASTIC PROGRAMMING

by

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SYSTEMS RESEARCH GROUP

A. Charnes, Director

CRITICAL PATH ANALYSES VIA CHANCE CONSTRAINED AND STOCHASTIC PROGRAMMING*

1. Introduction:

A question which combines statistics and linear programming considerations was first raised by G. Tintner in [11].1/ It concerns the distribution of optimum functional values when a linear programming problem has probabilistic constraints.

We propose to accord a chance constrained programming formulation to this kind of problem and to deal with it in a way that bears on project scheduling of the kind that is usually associated with critical path analysis.2/ For instance in PERT -- and related versions of critical path scheduling -- an attempt is made to deal with random time variations by reference to a procedure like the following. Three times are assumed for each task: (1) a pessimistic time value, (2) a normal time and (3) an optimistic time. These are multiplied by, respectively, 1/6, 2/3 and 1/6 and summed to derive a new time for each task. The new times are then used to derive a critical path from which an estimate of the total time is then derived.

In the present paper we shall assume that a particular distribution of times applies to each branch of the project network. Then we attempt to characterize the resulting distribution of total project completion times. This is done by virtue of a minimizing principle which implicitly carries with it a characterization of the critical paths that are associated with each set of time realizations that the distributions

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1/ See also [10].

2/ See [1], [6], [7], [8] and [9] as well as the references cited therein.

* We are indebted to Dr. D. Learner, and others on the staff at Batten, Barton, Durstine and Osborn for numerous discussions, assistance and encouragement on various aspects of this and related work.

Note: The material in this report provided the basis for one part of a talk presented by the authors at The Symposium on Mathematical Programming, Chicago, Assoc. for Computing Machinery and USAF Project RAND, June 19, 1962

admit for each branch of the network. We do not deal with these critical paths in any detail here, however, but reserve that topic for treatment in a subsequent paper. Only the simplest kind of decision rule--i.e., the zero order rule of chance constrained programming--will be exhibited here only for illustration to show how it encompasses the PERT rule as a special instance of a general class of risk control and evaluation procedures.

The main focus of this paper is on the statistical distributions of the project completion (and sub-completion) times. The question of total time distributions that we deal with can therefore be given a managerial policy flavor by assuming that, ab initio, a management is considering a contract for a certain project. The task sequences are known but the times are not known except in probability. Before contracting for a target completion date--with resulting delay penalties--this management would like to know the likely distribution of total times in order to decide whether to accept an offered contract or else bargain further on the completion dates, penalty rates and progress payments and prices.

2. Direct and Dual Linear Programming Problems for a Project Graph:

With known task times, t_j, the critical path scheduling problem may be formulated as a network flow problem

$$\max \sum_{j=1}^{n} t_j x_j$$

subject to

$$\sum_{j=1}^{n} \varepsilon_{ij} x_j = a_i \tag{1}$$

$$x_j \geq 0$$

$$i=0, \ldots, m$$

where $a_o = -1$, $a_n = 1$ and all other $a_i = 0$. The $x_j \geq 0$ corresponds to uni-directional time progression; the ε_{ij} are the incidence numbers for the network.1/

This is a linear programming problem. Hence it has a dual

$$\min \sum_{i=0}^{m} u_i a_i$$

(2) subject to

$$\sum_{i=0}^{m} u_i \varepsilon_{ij} \geq t_j .$$

In fact this dual provides the basis for an efficient computational algorithm--e.g., of the sub-dual algorithm variety.2/ Moreover, the value $\Sigma u_i a_i$ for any u_i satisfying the constraints of (2) provides an upper bound for the completion time associated with the critical path and, hence, associated also with the project's completion. Furthermore, the structure of the dual problem is such that for any given set of t_j, the optimal u_i may be determined by one pass through the network associated with (1).

Suppose that the known t_j's are now replaced by task times which are random variables with known statistical distributions. Because of the property of the dual (2)--and the related sub-dual algorithm3/--it is possible to compute, successively, u_i values which are minimal with

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1/ See [4] for further details or, more generally, see Chapter XVII in [3].

2/ See [3] and references cited therein. See also [1].

3/ See [1] and [3].

respect to the sub-graph constraints to which they apply. Thus when the t_j values are random variables these u_i values will also emerge as random variables. This means that, in principle, it is possible to determine the distributions of the u_i's in terms of the distributions of the t_j's which are constraining for these u_i values on the project graph.

The matter may be put another way by assuming that one starts back from the node associated with the project's completion. Moving back up the graph one then determines successive u_i values on a maximal time path to the associated node. Then, for fixed times, one applies to these u_i values the probability associated with each of the t_j times on the path leading up to this node. In this fashion one can obtain probabilities for the different time values corresponding to the u_i values achieved via the possible maximal time paths.

3. The Zero Order Rule:

The simplest of the chance constrained programming decision rules is a linear rule in which the coefficients of the t_j variables are set equal to zero.[1/] This means that we seek a set of values u_i which solve

$$\min \sum_{i=0}^{m} u_i a_i$$

(3) subject to

$$P\left(\sum_{i=0}^{m} u_i \varepsilon_{ij} \geq t_j\right) \geq \beta_j ,$$

1/ Vide, e.g., [2].

where "P" means "probability" and $0 \leq \beta_j \leq 1$ is a preassigned probability measure for $j=1, 2, \ldots, n$.

A solution of this problem amounts to determining a set of times associated with the inauguration of various tasks in such a manner that the probability of being able to complete the $j^{\underline{th}}$ task is at least β_j. The latter may be called the risk guarantees desired for this $j^{\underline{th}}$ task.

The left hand side of each chance constraint $j=1, \ldots, n$ can be rephrased analytically to achieve an expression in which $\Sigma\, u_i\, \varepsilon_{ij}$ is a parameter. Thus, if F_j is the marginal distribution of t_j we may write

$$(4)\qquad P(t_j \leq \sum_{i=0}^{m} u_i\, \varepsilon_{ij}) = F_j\, (\sum_{i=0}^{m} u_i\, \varepsilon_{ij})\ .$$

The chance constraint is thus transformed into a deterministic equivalent

$$(5)\qquad F_j\ (\sum_{i=0}^{m} u_i\, \varepsilon_{ij}) \geq \beta_j$$

or, in view of the monotonicity of F_j ,

$$(6)\qquad \sum_{i=0}^{m} u_i\, \varepsilon_{ij} \geq F_j^{-1}\ (\beta_j)\ .$$

Via this mode of development we have the following

$$(7)\qquad \begin{gathered} \min \sum_{i=0}^{m} u_i\ a_i \\ \text{subject to} \\ \sum_{i=0}^{m} u_i\, \varepsilon_{ij} \geq F_j^{-1}\ (\beta_j) \\ j=1, \ldots, n \end{gathered}$$

as a deterministic equivalent for (3). The number $F_j^{-1}(\beta_j)$ represents, of course, the fractile associated with β_j. Furthermore, this deterministic equivalent has a dual which exhibits, perhaps somewhat better, the character of the problem since it is in direct network form. This problem is

$$\max \sum_{j=1}^{n} F_j^{-1}(\beta_j)\, x_j$$

subject to

$$\sum_{j=1}^{n} \varepsilon_{ij}\, x_j = a_i \tag{8}$$

$$i=0, \ldots, m$$

$$x_j \geq 0.$$

It may be instructive to observe how the $F_j^{-1}(\beta_j)$ coefficients, in the functional, appear under some of the statistical distributions that might be assumed. If $F_j = N\ (\mu_j, \sigma_j)$ then

$$F_j^{-1}(\beta_j) = \mu_j + \sigma_j N^{-1}(\beta_j) \tag{9}$$

where N^{-1} is the inverse of the N(0, 1) distribution function. The functional of (8) then becomes

$$\sum_{j=1}^{n} F_j^{-1}(\beta_j)\, x_j = \sum_{j=1}^{n} (\mu_j + \sigma_j N^{-1}(\beta_j))\, x_j\ . \tag{10}$$

Another example is the log-normal distribution which gives the objective function

$$(11) \qquad \sum_{j=1}^{n} F_j^{-1}(\beta_j)\, x_j = \sum_{j=1}^{n} \left(e^{(\mu_j + \nu_{qj}\,\sigma_j)}\right) x_j$$

where e^{μ_j} is the median of the j^{th} log-normal distribution and ν_{qj} is the q^{th} fractile of $N(0, 1)$.

The PERT rule corresponds to using the population mean of the three valued random variable t_j with probabilities of, respectively, $p_{j\ell} = \frac{1}{6}$, $p_{jn} = \frac{2}{3}$, $p_{jh} = 1/6$. In general, for this distribution, the mean will not be an actually realized time. The protection level β_j, associated with this mean depends critically on whether the difference between the optimistic and normal time is greater or less than the difference between the normal time and the pessimistic time. The first case gives a protection level $\beta_j < 1/2$; the second case gives $\beta_j > 1/2$. When the two are equal, so that mean coincides with the mode and the median, then $\beta = 1/2$ will obtain.

With the above formulation we are thus able to relate the functional used to the risk levels desired. The PERT procedure, when the assumed distribution applies, also carries this feature with it. On the other hand, it is there presented in the guise of an estimating procedure which has an evident implication for the risk protection that is secured upon effecting the planning decisions. Thus, in particular, the choice of symmetric low and high estimates carries an implication that the actually realized times will, by random occurrences alone, lie either above or below the assumed value virtually all of the time.

4. Determination of Distribution of Times:

We now turn to the task of determining the critical path node variables for each possible set of values that the t_j variates may assume. That is, we want to determine implicitly the critical path for every possible collection of t_j realizations with their associated probabilities. We shall do this indirectly, however, in that we determine the distribution of the $w_i^* (t_1, \ldots, t_n)$ satisfying

$$\min \sum_{i=0}^{m} w_i a_i$$

with

$$\sum_{i=0}^{m} w_i e_{ij} \geq t_j \tag{12}$$

$$j=1, \ldots, n .$$

To compute the distributions for the w_i, we need have recourse only to obtaining the distribution function for sums of known random variables and for the maximum of a finite number of known random variables. For some distributions--e.g., finite and discrete distributions, gamma distributions, etc.--it is possible to carry out these operations explicitly. This may be an onerous task, of course, but the electronic computer may offer relief in cases like discrete distributions. Thus, in these particular kinds of cases, it is possible to develop the related distribution functions directly and thereby answer the question originally posed by Tintner [11].

To conclude this section we shall present an example of such a calculation. Consider the graph shown in Figure 1. The times of the jobs are marked alongside the arrow representing the job.

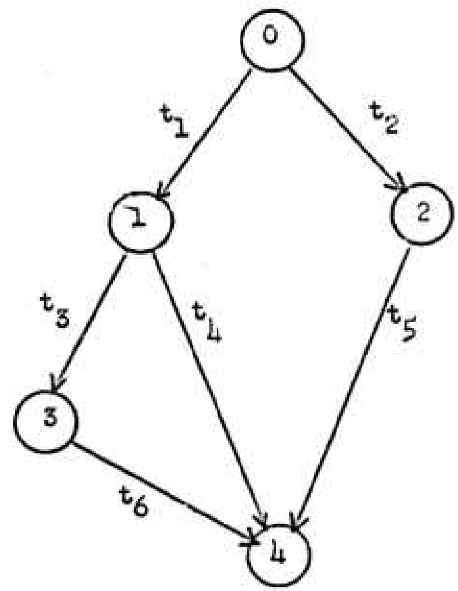


FIGURE 1

For this example the inequalities of (12) become

$$
\begin{array}{rrrrrcl}
w_1 & & & & & \geq & t_1 \\
 & w_2 & & & & \geq & t_2 \\
-w_1 & & +w_3 & & & \geq & t_3 \\
-w_1 & & & +w_4 & & \geq & t_4 \\
 & -w_2 & & +w_4 & & \geq & t_5 \\
 & & -w_3 & +w_4 & & \geq & t_6 \\
 & & & & w_o & = & 0
\end{array} \qquad (13)
$$

Since the solution of (12) is equivalent to finding the longest path through the network, the distribution of completion times can be found by successively finding the distribution of two jobs in series as in Figure 2(a), or in parallel as in Figure 2(b).

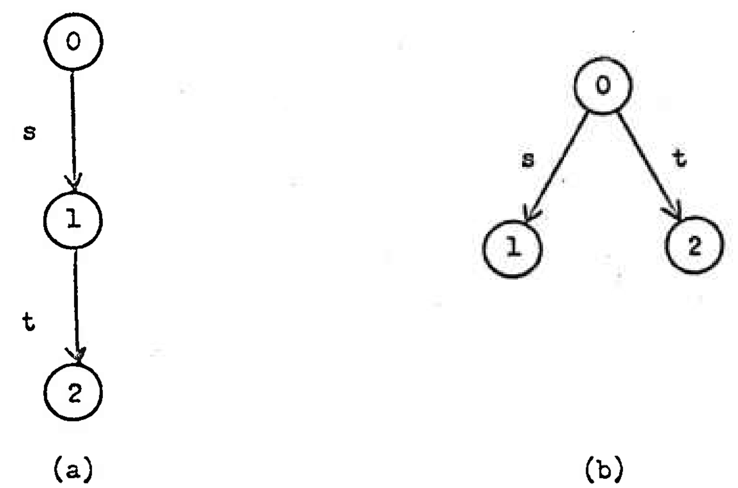


FIGURE 2

Clearly in (a), the time to complete both jobs is the sum s+t of the random variables; and in (b), the time to complete both jobs is the maximum of the two random variables. It is easy to establish the following two rules.

(A) The density of the sum of two independent random variables is the convolution of their densities.

(B) The distribution of the maximum of two independent[1/]

1/ For the critical path inequality structure it is possible to reduce all calculations to the use of these two rules even though at some stages one wants the maximum of two stochastically dependent random variables. See the example below.

random variables is the product of their two distributions.

In order to carry out the calculations explicitly we assume that all job times have an exponential density function

$$\theta\, e^{-\theta t}\;,\; 0 \leq t < \infty, \tag{14}$$

which has a distribution function

$$F(a) = P(t \leq a) = 1 - e^{-a\theta}\;. \tag{15}$$

If we now make the further simplifying assumption that the exponential functions are all characterized by the same parameter, θ, then we can tabulate the densities and distributions of the various w_i as shown in Table 1.

The calculations needed to obtain the entries of Table 1 are all elementary, but somewhat tedious. Hence we have tabulated them in detail so that this work need not be repeated by others. The same kinds of calculations could also be carried out for project graphs of bigger size, but the answers resulting might be very lengthy, perhaps several pages long, and, in general recourse to electronic computation would then be required. Pending completion of the instant codes, however, we can only conclude that an "in principle" demonstration of these possibilities has now been made.

TABLE 1

Variable	Random Variable	Density Function	Distribution Function
w_o	0		
w_1	t_1	$\theta e^{-\theta t}$	$1 - e^{-\theta t}$
w_2	t_2	$\theta e^{-\theta t}$	$1 - e^{-\theta t}$
w_3	$t_1 + t_3$	$\theta^2 t e^{-\theta t}$	$1-e^{-\theta t}(1+\theta t)$
	$t_2 + t_5$	$\theta^2 t e^{-\theta t}$	$1-e^{-\theta t}(1+\theta t)$
	$t_3 + t_6$	$\theta^2 t e^{-\theta t}$	$1-e^{-\theta t}(1+\theta t)$
w'	$\text{Max}\begin{cases} t_3 + t_6 \\ t_4 \end{cases}$	$\theta e^{-\theta t}[(1+\theta t) - e^{-\theta t}(1+2\theta t)]$	$1-e^{-\theta t}(1-e^{-\theta t}(1+\theta t))$
w''	$t_1 + \text{Max}\begin{cases} t_3 + t_6 \\ t_4 \end{cases}$	$\theta e^{-\theta t}[-3+\theta t + \frac{(\theta t)^2}{2} + (3+2\theta t)e^{-\theta t}]$	$1-e^{-\theta t}[-1+2\theta t + \frac{1}{2}(\theta t)^2 + e^{-\theta t}(2+\theta t)]$
w_4	$\text{Max}\begin{cases} t_2 + t_5 \\ t_1 + \text{Max}\begin{cases} t_3+t_6 \\ t_4 \end{cases} \end{cases}$	$\theta e^{-\theta t}\left\{ [-3+2\theta t + \frac{1}{2}(\theta t)^2] + e^{-\theta t}[6+5\theta t - \frac{7}{2}(\theta t)^2 - (\theta t)^3] - e^{-2\theta t}[3+7\theta t+4(\theta t)^2] \right\}$	$[1-e^{-\theta t}(1+\theta t)][1 - e^{-\theta t}(-1+2\theta t + \frac{1}{2}(\theta t)^2 + e^{-\theta t}(2+\theta t))]$

In order to get a better idea of the values of the functions in Table 1, we computed[1/] then using an electronic computer for various values of completion time and have plotted the results in Figures 3 and 4. Note in Figure 4 that the density function for w_4 is bi-modal, although the area under the first loop of its graph is so small that it does not appear on the corresponding distribution in Figure 3, the extremely flat approach to zero of the distribution for w_4 is especially evident in Figure 3. These and other features are displayed on the following charts -- which are based on electronic computations arranged for values of t running from 0 to 10 in increments of .02, to yield a mesh of 500 equally spaced points.[1/]

1/ The authors are indebted to Messrs. F. K. Levy and J. D. Wiest for carrying out these calculations on Carnegie Tech's Bendix G-20, and for drafting the accompanying charts, conducting supplementary analyses, etc.

Figure 3

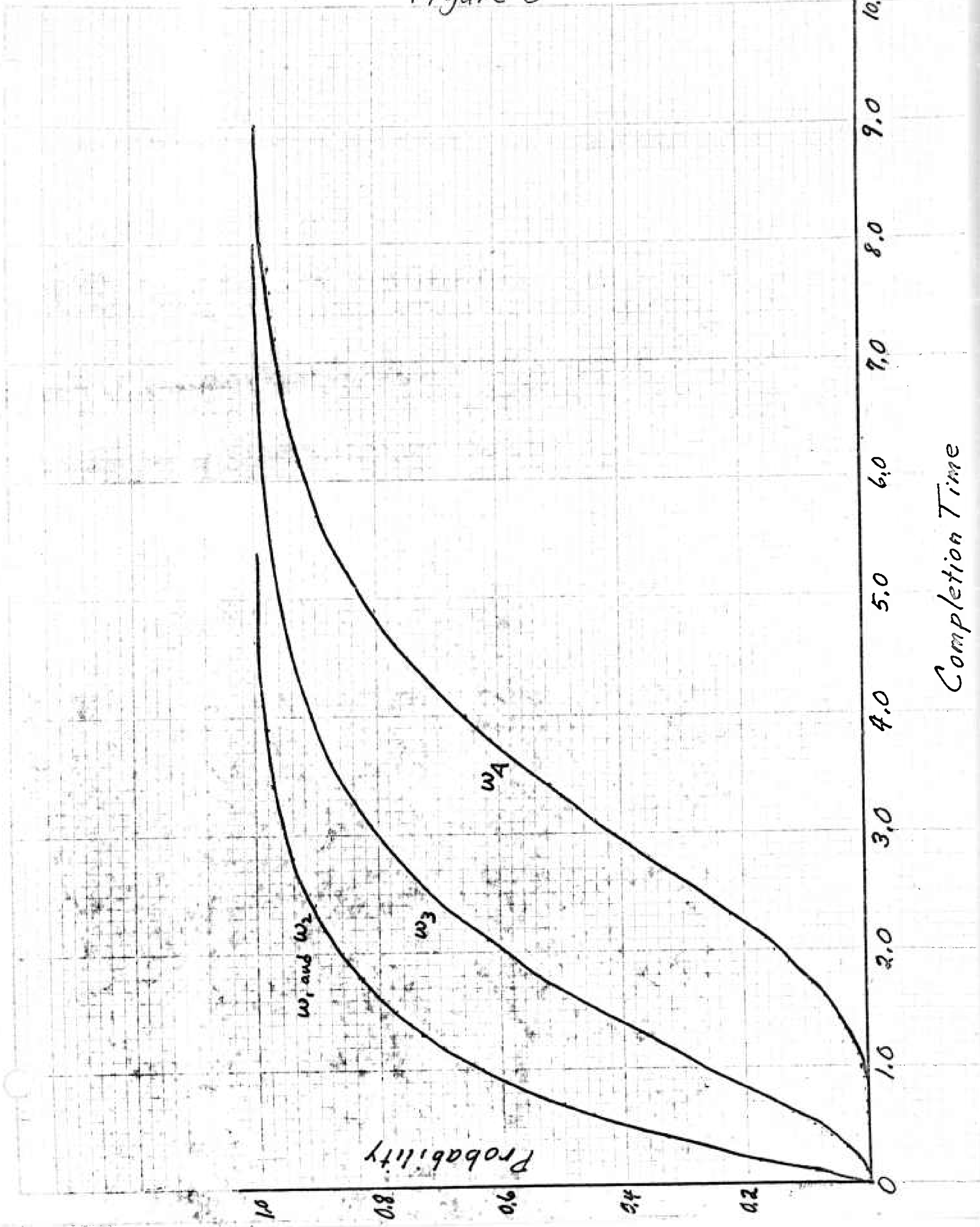
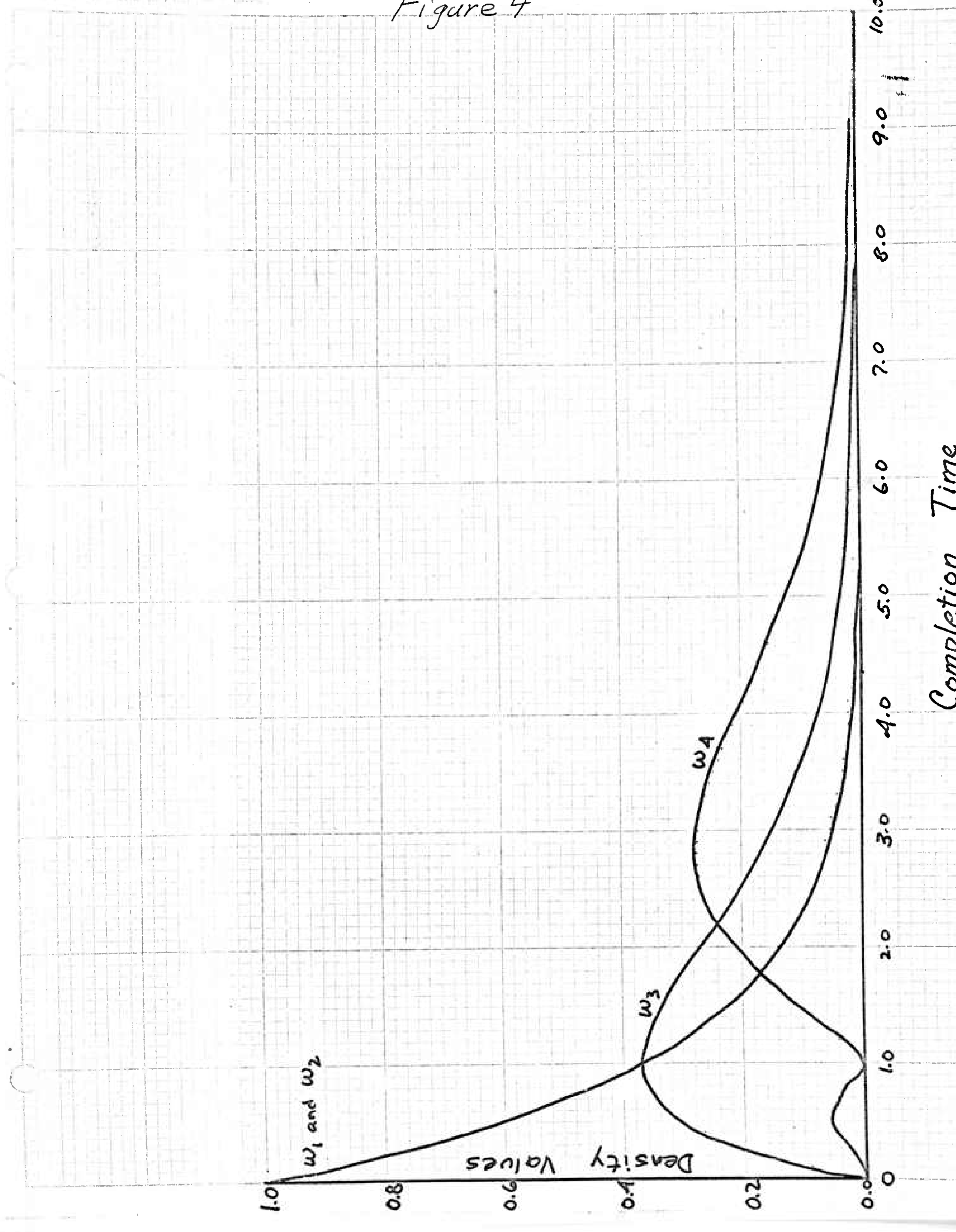

Probability
0.2
0.4
0.6
0.8
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
Completion Time
w_1 and w_2
w_3
w_4

Figure 4



Density Values
1.0
0.8
0.6
0.4
0.2
0.0
Completion Time
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
w_1 and w_2
w_3
w_4

5. Conclusion and Further Extensions:

On the basis of this highly simplified example we can draw some interesting (if tentative) conclusions. Note, first, that this exponential distribution has the property that each modal task time is zero. Furthermore, the function is monotone decreasing. Hence, larger completion times are less probable than short ones. Thus, this distribution would tend to produce higher probabilities for shorter overall completion times than other kinds of less optimistic distribution functions that might be specified. Nevertheless, asymptotic expansion shows that the overall distribution--the one for w_4--vanishes to the sixth order for $t=0$. I.e., a short overall completion time is highly improbable. Thus, the interacting properties of the graph relations produces a rather striking result for these random variations. The overall behavior is different than the behavior of each of the parts. It must be expected that other densities can display other equally striking (possibly even surprising) results.

One route of extension that is evidently open involves exploring other kinds of distributions and also, obviously, other kinds of project graphs. Computer codes for this purpose are in the process of being developed.

In the present case we have emphasized the distribution of early start times for all jobs. Then when the final node is achieved the time that is there attained becomes the early completion time (in the usual usage) for the entire project. Of course, by working backwards,

up the graph, we can effect the same kind of result for the so-called late start times. The difference between these two represents slack; but here we must allow for the possibility that negative as well as positive values of these variables may emerge. The probability of this happening may, of course, be computed. In fact this probability can then be accorded the status of a risk measure of departing from any possible critical path.

In current practice (e.g., PERT, etc.), the analysis is directed toward an explicit designation of a critical path for an entire project. But extensions via the higher order[1/] decision rules of chance constrained programming would suggest a somewhat different course. Early start times could then be utilized, for example, to provide effective procedures for developing multiple (dynamic) critical paths that take into account the deviations from anticipated task times, the precedence relations and the overall statistics of task time performance. This would include both risk and quality levels -- and associated evaluations and controls -- of the kind which chance constrained programming was explicitly designed to handle.[2/] It could also include non-independence of earlier and later task times, especially when aspects of task learning are involved on various parts of a project graph.[3/]

1/ I.e., of higher order than the ones discussed in <u>supra</u>, section 3. See, e.g., [2].

2/ See [5].

3/ Such "learning" or "progress curve" functions have proved to be of considerable importance in many cases -- e.g., as in setting contract terms and schedules for construction and new item production in World War II.

A use of higher order decision rules with a focus on early start times makes it possible to bypass the need for an explicit development of the resulting conditional critical paths (or portions thereof). That is, instead of specifying these paths *in toto*, as in current practice, the higher order decision rules of chance constrained programming would follow a different course of development. For instance, the emergence of specific early start times would be used to complete the certainty equivalent relations for chance constrained programming in order thereby to designate explicitly the next portion of the (implied) overall critical path.1/ Each of several conditional critical paths would then be implicitly carried forward and the final total chain (critical path) would then be known with certainty only upon the project's completion. Finally, the dual evaluators (when available) can be used to examine possible variations in risk or quality levels or, alternatively, the desirability of undertaking the total commitment implied by the contract can be ascertained by considering the statistical variations for minimum completion times which are likely to result from a given project graph whose links are subject to known random time variations.

1/ *Vide*, e.g., [2].

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